An inequality concerning the production of vorticity in isotropic turbulence

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SUMMARY

An inequality is demonstrated involving the rate of production S of mean-square vorticity in isotropic turbulence and a factor γ which may be said to allow for intermittency or for the non-vanishing of fourth-order cumulants. An extreme state, corresponding to equality of this relationship, occurs if $S = 0{\cdot}64$ and $\gamma = 2{\cdot}14$. The experimental values are $S = 0{\cdot}4$ and $\gamma = 4$. Another kinematical relation shows that the mechanism of vorticity production resembles collision between fluid particles rather than the swirling of contracting jets.

1. INTRODUCTION

When the motion of a fluid is turbulent, the fluctuations of the vorticity are of major importance. For isotropic conditions they are directly related to energy dissipation by viscosity. They also show a sufficient degree of isotropy and homogeneity in general to justify the corresponding simplifications of the theory.

These fluctuations obey a system of equations in which the pressure does not appear and thereby the degree of complication is reduced. According to these equations, viscosity tends to eliminate vorticity, by a well-understood process. These equations also show that the mean-square vorticity can be increased by inertial effects. This aspect of the problem is still somewhat mysterious. What determines this production of vorticity and how does it occur?

The problem has been stated in terms of correlation functions, but this procedure leads to a number of unknowns always larger than the number of equations.

The Fourier transforms of the correlations can be introduced, amounting to a spectral analysis of a sample of turbulent flow. They lead to a relation between vorticity production and the rate of energy transfer from large eddies to small eddies. Each hypothesis concerning energy transfer has thus been followed by an evaluation of the rate of production of vorticity.

In this paper, we investigate the largest rate of production of vorticity compatible with the requirements of isotropy, homogeneity and incompressibility. We find an upper bound for the rate of vorticity production, although it depends upon a fourth-order mean value and therefore does not constitute an absolute maximum. Comparison with old and new experimental data indicates that a turbulent flow produces vorticity at about 50% of this limiting rate. It is quite possible that dynamical effects are responsible for this factor and that we are confronted by an extreme state of affairs, characteristic of turbulent flows.

2. Fundamental inequality

We intend to establish an inequality valid at every point of an incompressible flow and at every instant. We shall use cartesian coordinates x_i, denote the velocity vector by u_i, and write $u_{i,j}$ for $\partial u_i/\partial x_j$. The condition of incompressibility, with the usual convention about summation of repeated indices, is

$$u_{j,j} = 0. \tag{1}$$

We can write $u_{i,j} = s_{ij} + r_{ij}$, where s_{ij} is the symmetric rate of strain tensor and r_{ij} the antisymmetric vorticity tensor. At any point and instant, we can find the principal axes of s_{ij} and, on referring to these axes, we have:

$$u_{i,j} = \begin{vmatrix} a & C & -B \\ -C & b & A \\ B & -A & c \end{vmatrix}, \tag{2}$$

where a, b, c are the eigenvalues of s_{ij} and A, B, C are half the vorticity components in the directions of the principal axes. In the sequel we shall designate the directions associated with a, b, c as the a-axis, b-axis and c-axis. Furthermore, we shall adopt the convention

$$|a| \geqslant |b| \geqslant |c|. \tag{3}$$

By (1) we have

$$a+b+c = 0, \tag{4}$$

and consequently b and c must be of the same sign and a and b of opposite sign.

We shall make use of the general Cauchy inequality (Hardy *et al.* 1952)

$$(a^3+b^3+c^3)^2 \leqslant (a^2+b^2+c^2)(a^4+b^4+c^4) \tag{5}$$

with equality if $a = b = c$. We shall also use the following identities, obtained by raising both sides of (4) to powers 2, 3 and 4,

$$\left.\begin{aligned} a^2+b^2+c^2 &= -2(ab+bc+ca), \\ a^3+b^3+c^3 &= 3abc, \\ a^4+b^4+c^4 &= \tfrac{1}{2}(a^2+b^2+c^2)^2. \end{aligned}\right\} \tag{6}$$

Values of a, b and c for which the two sides of (5) are nearly equal are excluded by the incompressibility condition. Indeed, when a, b and c satisfy (4), the ratio of the left-hand side to the right-hand side of (5) has a maximum value of $\frac{1}{3}$ at $-a = 2b = 2c$. This allows a factor $\frac{1}{3}$ to be inserted on the right-hand side of (5), and so, with the use of (6), we have

$$|abc| \leqslant \frac{1}{3\sqrt{6}}(a^2+b^2+c^2)^{3/2}. \tag{7}$$

It is known that the set of eigenvalues of a tensor is invariant to transformations of the coordinates; therefore each term of (7) is invariant. Indeed, each term can be expressed by invariants such as $s_{ij}s_{ij}$ or $s_{ij}s_{jk}s_{ki}$. Consequently (7) remains valid if we pass to a fixed system of coordinates, independent of the orientation of the a-, b-, and c-axes. We now consider an ensemble of points of the fluid, each with its own set of eigenvalues with respect to these fixed axes, and average both sides of (7). We can use a space, a time or an ensemble average. Then, with $\langle \; \rangle$ indicating the average, we have

$$|\langle abc\rangle| \leqslant \frac{1}{3\sqrt{6}} \langle (a^2+b^2+c^2)^{3/2}\rangle. \tag{8}$$

The right-hand side of (8) cannot be measured by conventional methods and comparison with experimental results is not possible unless we introduce a modified inequality. We can write a single inequality involving N sets of eigenvalues, viz.

$$\left(\sum_{n=1}^{N} a_n^3+b_n^3+c_n^3\right)^2 \leqslant \left(\sum_{1}^{N} a_n^2+b_n^2+c_n^2\right)\left(\sum_{1}^{N} a_n^4+b_n^4+c_n^4\right) \tag{9}$$

By taking (4) and (6) into account, and introducing the average as a limit for large N, the inequality (9) becomes

$$|\langle abc\rangle| \leqslant \frac{1}{3\sqrt{3}} \langle a^2+b^2+c^2\rangle^{1/2}\langle a^4+b^4+c^4\rangle^{1/2}. \tag{10}$$

The right-hand side of (8) is always less than or equal to the right-hand side of (10), with equality if a, b and c do not fluctuate. If, in addition, $a = -2b = -2c$, the two sides of (10) are equal. In view of (6), the ratio of right-hand sides of (8) and (10) depends only upon the probability distribution of $a^2+b^2+c^2$. In the case of a Gaussian probability distribution of $a^2+b^2+c^2$, the right-hand side of (8) is only 8% smaller than that of (10). In general, this ratio is close to unity, unless the probability distribution has a pronounced peak at the origin. Thus, the inequality (10) is not much weaker than (8).

3. Relations with measurable quantities

The various invariants occurring in (10) can be measured with a single hot-wire anemometer. This has been done in the turbulence produced by a grid obstructing a parallel flow of air. The hot-wire is located sufficiently far down-stream to give some guarantee of isotropic conditions, and its orientation is normal to the mean flow. The turbulence passes by the wire with a mean velocity about one hundred times larger than the velocity fluctuations, and consequently the time derivative of the hot-wire signal corresponds reasonably well to the space derivative of the velocity. With index 1 denoting the direction of mean flow, this means that the differentiated signal is proportional to $u_{1,1}$.

Let us find the relation between $u_{1,1}$ and the invariants of (10). At any particular instant, the hot-wire responds to a fluid particle whose principal axes of rate of strain can be specified. Let ϕ be the angle between the 1-axis and the a-axis (latitude) and ψ be the angle between the c-axis and a direction normal to the a-axis and 1-axis (longitude). It then follows from tensor calculus that

$$u_{1,1} = a\cos^2\phi + b\cos^2\psi\,\sin^2\phi + c\sin^2\psi\sin^2\phi. \tag{11}$$

The vorticity does not contribute to this particular signal, but this is peculiar to the component $u_{1,1}$,

In the course of time the hot-wire is coincident with different fluid particles and the angles ϕ and ψ vary as well as the eigenvalues a, b, c. The eigenvalues are absolute quantities. Consequently, if the turbulence is isotropic, the angles must be statistically independent of the eigenvalues. To clarify this point let us consider all fluid particles for which $a = 1$, $b = -0{\cdot}7$, $c = -0{\cdot}3$. If, in the average over these particles, we find a definite orientation of the principal axis, we can use the numbers 1, $-0{\cdot}7$, $-0{\cdot}3$ to specify a direction. Such a direction would remain invariant to a rotation of the coordinates and this would reveal a lack of isotropy. Therefore, we must admit that, for any set of values of a, b, c, the orientation of the principal axis are random; that is, we have statistical independence of eigenvalues and orientations.

With $(1/4\pi)\sin\phi\,d\phi\,d\psi$ as the probability of finding ϕ and ψ inside a small solid angle, and statistical independence, we find

$$\langle u_{1,1}^2\rangle = \frac{2}{15}\langle a^2+b^2+c^2\rangle. \tag{12}$$

The same method can be used to demonstrate

$$\langle u_{1,1}^3\rangle = \frac{24}{105}\langle abc\rangle, \tag{13}$$

$$\langle u_{1,1}^4\rangle = \frac{8}{105}\langle a^4+b^4+c^4\rangle. \tag{14}$$

Some of these results have been derived by others from correlation functions. When the following non-dimensional parameters are introduced,

$$S = -\frac{\langle u_{1,1}^3\rangle}{\langle u_{1,1}^2\rangle^{3/2}}, \qquad \gamma = \frac{\langle u_{1,1}^4\rangle}{\langle u_{1,1}^2\rangle^2}, \tag{15}$$

the inequality (10) can be expressed as:

$$|S| \leqslant \frac{2}{\sqrt{21}}\,\gamma^{1/2}. \tag{16}$$

For normally distributed velocities we have $\gamma = 3$, and in that case $|S| \leqslant 0{\cdot}756$.

In the extreme case $-a = 2b = 2c =$ constant, we find by (12), (13) and (14), $\gamma = 2{\cdot}14$, $|S| = 0{\cdot}638$. Proudman & Reid (1954) calculated the value of S with the assumptions of no viscosity, zero initial triple correlations and a relation between three-point quadruple velocity correlations and double correlations appropriate for normal probability distribution. They found $S = 0{\cdot}78$, and by (16) this implies $\gamma \geqslant 3{\cdot}2$. This seems to mean that the four-point quadruple correlations could not satisfy exactly a similar relation with double correlations, since this would imply $\gamma = 3$.

4. Experimental results

The numerous measurements of S and γ by Townsend (1947, 1948, 1951) gave $S = 0{\cdot}38$, $\gamma = 4$. Later Stewart (1951) found values of S decreasing with increasing Reynolds number of the turbulence, in the range $S = 0{\cdot}5$ to $0{\cdot}3$. From the measurements of $\langle \partial^2 u_1/\partial x_1^2 \rangle$ by Liepmann *et al.* (1951) one can find S, by making use of the vorticity equation. The resulting values of S increase from $0{\cdot}4$ to $0{\cdot}5$ with increasing Reynolds numbers.

The author measured S and γ for a variety of meshes and with improved electronic devices in the low-turbulence wind tunnel of the University of Maryland. The hot-wire signal was carefully compensated for thermal inertia. A simple low-pass filter was used to avoid excessive electronic noise, and it was verified that the filter did not affect the measurements of mean cubes.

To measure mean squares and mean cubes, I used chains of biased diodes. For mean fourth power, I first obtained the instantaneous square with a quarter-square multiplier using four chains of diodes.

The first grids used were of 2·5 cm mesh, with 38 and 66% open area and air velocities of 2 to 20 m/sec. The wire was located at distances from 30 to 100 mesh lengths from the grid. The results consistently indicated $S = 0{\cdot}4 \pm 0{\cdot}1$, without systematic variation.

With a grid of 0·6 cm mesh, 21% open area and a wind of 4 m/sec I found $S = 0{\cdot}4 \pm 0{\cdot}05$ at distances of 20 to 150 mesh lengths. With a first grid of 2·5 cm and 54% open area, followed at 125 cm downstream by a second grid of 1·25 cm and 76% open area, I found $S = 0{\cdot}4$ at a distance of 200 cm from the first grid. When several strips of paper tape were added to the first grid the turbulence was considerably altered thereby, but S was found to be unchanged.

In a crude pipe flow and in the wake behind a cylinder, I again found $S = 0{\cdot}4$. The central part of a turbulent boundary layer gave the same result.

Only a few measurements of γ were made, and they indicate $\gamma = 4 \pm 0{\cdot}5$.

The constancy of S is remarkable, and holds at Reynolds numbers lower than those for which Kolmogoroff's theory might be expected to be valid. Comparison with the inequality (16) shows that the experimental value of $S/\gamma^{1/2}$ corresponds roughly to 50% of the maximum permissible value. With $\gamma = 4$, the inequality requires $|S| \leqslant 0{\cdot}872$.

5. Production of vorticity

We shall now introduce the dynamical equations. With additional indices to indicate successive derivatives, we write the Navier–Stokes equations as

$$\partial u_i/\partial t + u_k u_{i,k} - \nu u_{i,kk} = -P_i/\rho, \tag{17}$$

where ν is the kinematical viscosity, P the pressure and ρ the density. After differentiating with respect to x_j, multiplication by $u_{i,j}$, summation in i and j, and averaging we obtain

$$\frac{\partial}{\partial t}\langle u_{i,j}u_{i,j}\rangle+2\langle u_k u_{i,j}u_{i,kj}\rangle+2\langle u_{i,j}u_{i,k}u_{k,j}\rangle-$$
$$-2\nu\langle u_{i,j}u_{i,jkk}\rangle = -\frac{1}{\rho}\langle u_{i,j}P_{ij}\rangle. \quad (18)$$

For the first term we can write, with reference to (2),

$$\langle u_{i,j}u_{i,j}\rangle = \langle a^2+b^2+c^2\rangle+2\langle A^2+B^2+C^2\rangle. \quad (19)$$

We now consider the invariant $\langle u_{i,j}u_{j,i}\rangle$. In homogeneous and incompressible flow, we have

$$\langle u_{i,j}u_{j,i}\rangle = \frac{\partial}{\partial x_j}\langle u_i u_{j,i}\rangle = 0, \quad (20)$$

giving the following relation between the mean-square vorticity and the mean-square rate of strain:

$$\langle a^2+b^2+c^2\rangle = 2\langle A^2+B^2+C^2\rangle. \quad (21)$$

The invariant $\langle u_k u_{i,j}u_{i,jk}\rangle$ also vanishes, either by homogeneity or by isotropy. Also, for the same reasons, $\langle P_{ij}u_{i,j}\rangle = 0$. The viscous term can be related to the vorticity or the rate of strain, and can also be written as

$$\langle u_{i,j}u_{i,jkk}\rangle = -\langle u_{i,jk}u_{i,jk}\rangle. \quad (22)$$

The invariant $\langle u_{i,j}u_{i,k}u_{k,j}\rangle$ can be written as $\langle u_{i,j}u_{j,k}u_{i,k}\rangle$, where the sequence of indices must be noted. With the use of (2), it becomes

$$\langle u_{i,j}u_{j,k}u_{i,k}\rangle = \langle a^3+b^3+c^3\rangle-\langle aA^2+bB^2+cC^2\rangle. \quad (23)$$

Let us consider the invariant $\langle u_{i,j}u_{j,k}u_{k,i}\rangle$, where the index sequence is now different. This quantity vanishes because it can be expressed as the divergence of a vector;

$$\langle u_{i,j}u_{j,k}u_{k,i}\rangle = \frac{\partial}{\partial x_i}\langle u_{i,j}u_{j,k}u_k-\tfrac{1}{2}u_i u_{k,j}u_{j,k}\rangle = 0. \quad (24)$$

Equation (24) is a relation imposed by homogeneity, and with the use of (2) it becomes

$$\langle a^3+b^3+c^3\rangle = -3\langle aA^2+bB^2+cC^2\rangle. \quad (25)$$

This relation is comparable with (21) in that it is purely kinematical. The right-hand side describes the stretching of vortex lines, as discussed by Taylor (1938).

We can now use relations (19), (21), (22), (23), and (25) and rewrite the dynamical equation (18) as

$$\frac{\partial}{\partial t}\langle A^2+B^2+C^2\rangle = -\langle abc\rangle-\tfrac{1}{4}\nu\langle u_{i,jk}u_{i,jk}\rangle. \quad (26)$$

The viscous term is always positive and indicates dissipation of vorticity. Production of mean-square vorticity occurs if $\langle abc\rangle < 0$, which is equivalent to $S > 0$. Our inequalities therefore specify a maximum rate of production of vorticity with a limit determined by mean square and mean fourth powers of the eigenvalues.

Let us examine the effect of intermittency of the flow. We will say that the flow is intermittent if $u_{i,j} \equiv 0$ in a variety of large and well defined regions, occupying a fraction α of the space filled by the fluid. The distribution of these regions can be isotropic and homogeneous and they can eventually have a size comparable with the integral scale of the turbulence. The quantities S and γ can be measured as previously or similar quantities S' and γ' can be obtained by averaging only over those regions where $u_{i,j}$ fluctuates. It follows from (15) that $S' = \surd(1-\alpha)S$ and $\gamma' = (1-\alpha)\gamma$. Hence $S'/\surd\gamma'$ is independent of α, and this means that the presence of γ in (16) allows for possible intermittency of the flow.

The factor γ' is related to quadruple four-point correlations, and one could perhaps assume that, within the regions of finite $u_{i,j}$, the fourth order cumulants vanish. This would give $\gamma' = 3$. Observation of turbulent flows suggests that γ and *a fortiori* γ' are not very different from 3, and that $u_{i,j}$ is mildly intermittent. This means that S could not exceed a value of about 0·9 without serious intermittency or without non-zero fourth order cumulants.

6. A remark on vorticity production

From (6) and (25), we have the important relation

$$-\langle abc\rangle = \langle aA^2+bB^2+cC^2\rangle.$$

In a different form, this relation was first written by Townsend (1951). We must remember that b and c are of the same sign and that a is the largest eigenvalue. Thus abc has the sign of a and, for production of vorticity, we must have a predominance of points at which $a < 0$.

If $a < 0$, the flow behaves locally as a jet parallel to the a-axis and impinging on a wall represented by the b- and c-axes. The dynamic equations show that A (the vorticity component normal to the wall) is attenuated, and that B and C are amplified. If this situation lasts long enough, we shall find, for any initial vorticity, that $aA^2+bB^2+cC^2 > 0$. This situation is therefore such that both sides of (27) are positive. If $a > 0$, the flow behaves locally as if it were entering a contraction cone. A small fluid sphere is deformed into a cigar-shaped ellipsoid. Only A is amplified, and if this situation lasts long enough, it will contribute positively to the right-hand side of (27) and negatively to the left-hand side.

It is clear, therefore, that production of vorticity is associated essentially with $a < 0$ and amplification of B and C. This suggests that the most important processes associated with production of vorticity and energy transfer resemble a jet collision and not the swirling of a contracting jet.

7. Remark on continuity

So far, we have not mentioned any requirement of continuity, and the existence of second derivatives of the velocity has been assumed implicitly.

This question is of major interest, especially in the case of very small viscosity. Let us consider the extreme case of our inequality, that is, $a = -2b = -2c =$ constant. Equations (21) and (25) impose some restrictions on the vorticity, and it turns out that they involve only $\langle A^2 \rangle$ and $\langle B^2 + C^2 \rangle$. This leaves a great freedom in the choice of the vorticity and the orientation of the principal axes. However, this does not guarantee that we can connect smoothly the values of $u_{i,j}$ at a point with its values at some other point. When the viscosity is very small the momentum equation can conceivably include large values of $u_{i,kk}$, but there may be a limit to the type and magnitude of the viscous terms. This leads to the following question: Is it possible to construct a velocity field with maximum $\langle abc \rangle$, isotropic and homogeneous properties and such that it satisfies the momentum equations? In the limit of vanishing viscosity discontinuities of $u_{i,j}$ occur, and we can ask whether the condition $a = -2b = -2c =$ constant makes the flow everywhere discontinuous or whether we can choose the other components of $u_{i,j}$ in such a way that the discontinuities occur only along isolated surfaces (shear layers). The answer to such questions could very well lead to a lower maximum value of S and eventually close the gap between experiments and theory.

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